The normalized Ricci flow on some homogeneous spaces under a dynamical point of view

Andreas Arvanitoyeorgos
University of Patras, Greece

The talk is devoted to some recent developments about the dynamical behaviour of the normalized Ricci flow on *generalized Wallach spaces.* These are homogeneous spaces G/H whose isotropy representation $\mathfrak{p}$ decomposes into a direct sum of three $\mathrm{Ad}(H)$-invariant irreducible modules $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}$, with the property $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$. Despite their simple algebraic description these spaces were only very recently classified independently by Yu.G. Nikonorov in [Ni] and Z. Chen, Y. Kang and K. Liang in [Ch-Ka-Li]. The normalized Ricci flow equation $\dfrac{\partial}{\partial t}\mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n}$ is well known, introduced by R. Hamilton in 1982. For an n-dimensional generalized Wallach space endowed with a G-invariant Riemannian metric, the normalized Ricci flow reduces to a system of ODE's of the form

$$\frac{dx_1}{dt} = f(x_1, x_2, x_3), \quad \frac{dx_2}{dt} = g(x_1, x_2, x_3), \quad \frac{dx_3}{dt} = h(x_1, x_2, x_3), \tag{1}$$

where $x_i = x_i(t) > 0$, $i = 1, 2, 3$ are the parameters of the invariant metric,

$$f(x_1, x_2, x_3) = -1 - a_1 x_1 \left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right) + x_1 B,$$

$$g(x_1, x_2, x_3) = -1 - a_2 x_2 \left(\frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right) + x_2 B,$$

$$h(x_1, x_2, x_3) = -1 - a_3 x_3 \left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right) + x_3 B,$$

$B = \left(\frac{1}{a_1 x_1} + \frac{1}{a_2 x_2} + \frac{1}{a_3 x_3} - \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right) \right) \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)^{-1}$, and a_i, $i = 1, 2, 3$, are some real numbers in the interval $(0, 1/2]$. It can be checked that the volume $V = x_1^{1/a_1} x_2^{1/a_2} x_3^{1/a_3}$ is the first integral of the system (1). Therefore, we can reduce it to the system of two differential equations on the surface $V \equiv 1$ of the form

$$\frac{dx_1(t)}{dt} = \widetilde{f}(x_1, x_2), \quad \frac{dx_2(t)}{dt} = \widetilde{g}(x_1, x_2), \tag{2}$$

where $\widetilde{f}(x_1, x_2) = f(x_1, x_2, \varphi(x_1, x_2))$, $\widetilde{g}(x_1, x_2) = g(x_1, x_2, \varphi(x_1, x_2))$, $\varphi(x_1, x_2) = x_1^{-\frac{a_3}{a_1}} x_2^{-\frac{a_3}{a_2}}$.

It is easy to see that the singular points of the system (1) are exactly the invariant Einstein metrics on the generalized Wallach space under consideration, previously studied by E.V. Firsov, A.M. Lomshakov, Yu.G. Nikonorov in [Lo-Ni-Fi]. In this work it was shown that there at most four Einstein metrics (up to homothety) for every such space. In the present talk I will discuss the recent work [Ab-Ar-Ni-Si] where we studied the singular points of the system (1) (as Einstein metrics on generalized Wallach spaces). We gave a condition of degeneracy and analysed the types of singularities of singular points of the system (2), in a qualitative point of view. Furthermore, we partially studied a real algebraic surface $Q(a_1, a_2, a_3) = 0$ in $\mathbb{R}^3$ which contains all points (a_1, a_2, a_3) for which system (2) has at least one degenerate singular point. There were some difficult open problems left in our study (e.g. the topological properties of this surface $Q(a_1, a_2, a_3) = 0$) that were recently solved by N.A. Abiev in [Ab] and by A.B. Batkhin, A.D. Bruno in [Ba-Br]. I will present some new directions of study on this topic.

References

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